# Technical Comments

## Stress Compatibility Equations in Cylindrical Coordinates

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THE compatibility equations relating the components of strain in curvilinear coordinates are well-known.[1] Tuba, in a recent technical comment in this journal,[2] corrected Vlasov's[3] specialization of these equations to a plane orthogonal frame and their further reduction to the case of plane polar coordinates. His last equation [Eq. (5), Ref. 2] can readily be obtained from Brdička's earlier work[4] where these equations are given in cylindrical coordinates.

In stress analysis, especially when the surface tractions are specified, it is often more convenient to solve the stress rather than the displacement problem, for which it is necessary that the stress compatibility equations be satisfied. They are given here for reference purposes since a survey of the literature indicates that they are not available elsewhere;

$$(1+\nu)\nabla^2\tau_{xx} + \tau_{kk,xx} = -E\alpha\left(\frac{1+\nu}{1-\nu}\nabla^2 T + T_{,xx}\right) \times$$

$$(1+\nu)[\nabla^2\tau_{rr} - 4\tau_{r\theta,\theta}/r^2 - 2(\tau_{rr}-\tau_{\theta\theta})/r^2] + \tau_{kk,rr} = -E\alpha\left(\frac{1+\nu}{1-\nu}\nabla^2 T + T_{,rr}\right)$$

$$(1+\nu)[\nabla^2\tau_{\theta\theta} + 4\tau_{r\theta,\theta}/r^2 + 2(\tau_{rr}-\tau_{\theta\theta})/r^2] + \tau_{kk,r}/r + \tau_{kk,\theta\theta}/r^2 = -E\alpha\left(\frac{1+\nu}{1-\nu}\nabla^2 T + T_{,\theta\theta}/r^2 + T_{,r}/r\right) \quad (1)$$

$$(1+\nu)[\nabla^2\tau_{r\theta} + 2(\tau_{rr}-\tau_{\theta\theta})_{,\theta}/r^2 - 4\tau_{r\theta}/r^2] + (\tau_{kk,\theta}/r)_{,r} = -E\alpha(T_{,\theta}/r)_{,r}$$

$$(1+\nu)(\nabla^2\tau_{\theta x} + 2\tau_{rx,\theta}/r^2 - \tau_{\theta x}/r^2) + \tau_{kk,\theta x}/r = -E\alpha T_{,\theta x}/r$$

$$(1+\nu)(\nabla^2\tau_{rx} - 2\tau_{\theta x,\theta}/r^2 - \tau_{rx}/r^2) + \tau_{kk,rx} = -E\alpha T_{,rx}$$

where

$$\tau_{kk} = \tau_{xx} + \tau_{rr} + \tau_{\theta\theta}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

and where $\tau_{ij}$ are the physical components of stress, $E$ is the modulus of elasticity, $\nu$ is Poisson's ratio, $\alpha$ is the coefficient of thermal expansion, and $T$ is the temperature change from the quiescent state.

These equations reduce to those given by Lee[5] for the special case of both axial symmetry and temperatures varying only with the axial coordinate. In general, curvilinear coordinates these equations are

$$(1+\nu)\sigma_{ij|mn}g^{mn} + (\sigma_{mn}g^{mn})|_{ij} = -E\alpha\left(\frac{1+\nu}{1-\nu}T|_{mn}g^{mn}g_{ij} + T|_{ij}\right) \quad (2)$$

where the summation convention applies to a repeated index that represents covariance in one of the quantities in which it appears and contravariance in the other, and where $i, j, m, n$, can each take any of the values 1, 2, and 3; $\sigma_{ij}$ are the covariant components of the second-rank stress tensor; $|_i$ is a covariant derivative with respect to $x^i$; $g_{ij}$ are the covariant components of the metric tensor; and $g^{ij}$ are the components of the contravariant reciprocal of the metric tensor.

Equation 2 reduces to Brdička's [Ref. 4, Eq. (54) and Ref. 6, Eq. (7)] when $T$ is a linear function in an orthogonal Cartesian frame. It also reduces to that given by Boley and Wiener[7] for the case of an orthogonal Cartesian frame, and to Eq. (1) for the case of cylindrical coordinates.

Received October 24, 1967.
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### References

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[3] Vlasov, V. Z., "The Equations of Continuity of Deformations in Curvilinear Coordinates," *Prik. Mat. Mekhan.*, Vol. 8, 1944, p. 301.
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[5] Lee, C. W., "Thermoelastic Stresses in Thick-Walled Cylinders under Axial Temperature Gradient," *Transactions of the American Society of Mechanical Engineers*, Vol. 88, *Series E: Journal of Applied Mechanics*, Vol. 33, 1966, pp. 467–469.
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[7] Boley, B. A. and Wiener, J. H., *Theory of Thermal Stresses*, Wiley, New York, 1960.

## Comments on "Generalized Law of the Wall and Eddy-Viscosity Model for Wall Boundary Layers"

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IN a recent paper, Kleinstein[1] makes an analysis of the law of the wall for the hydraulically smooth case. Utilizing Prandtl's familiar mixing length theory in the form of an overlap layer viscosity function,

$$\epsilon^+ = k_1 u_\tau^+ y^+ \quad (1)$$

plus the cubic power sublayer variation theorized by Reichardt,[2]

$$\epsilon \sim y^3 \sim u^3 \quad (2)$$

Kleinstein derives a single formula for the law of the wall which is valid both in the sublayer and in the overlap layer:

$$y^+ = u^+ + (1/k_2)[\exp(k_1u^+) - 1 - k_1u^+ - \tfrac{1}{2}(k_1u^+)^2 - \tfrac{1}{6}(k_1u^+)^3] \quad (3)$$

Received August 11, 1967.
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